

12.4 The Cross Product

We define the cross product, or vector product, for two 3D vectors,

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \text{ and}$$

$$\mathbf{b} = \langle b_1, b_2, b_3 \rangle,$$

by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$= (a_2 b_3 - a_3 b_2) \mathbf{i} - (a_1 b_3 - a_3 b_1) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}$$

$$\text{Ex: } \mathbf{a} = \langle 1, 2, 0 \rangle \text{ and } \mathbf{b} = \langle -1, 3, 2 \rangle$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ -1 & 3 & 2 \end{vmatrix} =$$

$$(\quad - \quad) \mathbf{i} - (\quad - \quad) \mathbf{j} + (\quad - \quad) \mathbf{k}$$

You do: $\mathbf{a} = \langle 1, 3, -1 \rangle$, $\mathbf{b} = \langle 2, 1, 5 \rangle$.

Compute $\mathbf{a} \times \mathbf{b}$

Most important fact: (*HW 1-3, 7*)

The vector $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is
orthogonal to *both* $\mathbf{a}$ and $\mathbf{b}$.

Note: If **a** and **b** are parallel to each other, then there are many vectors perpendicular to both **a** and **b**.
So what happens to $\mathbf{v} = \mathbf{a} \times \mathbf{b}$? (HW 10)

Example: Give me any two vectors that are parallel and let's see.

Right-hand rule (HW 6, 10)

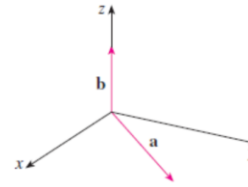
If the fingers of the right-hand curl from $\mathbf{a}$ to $\mathbf{b}$, then the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

Order matters: $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

1. [Question Details](#)

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The figure shows a vector $\mathbf{a}$ in the xy -plane and a vector $\mathbf{b}$ in the direction of $\mathbf{k}$. Their lengths are $|\mathbf{a}| = 6$ and $|\mathbf{b}| = 2$.



(a) Find $|\mathbf{a} \times \mathbf{b}|$.

(b) Use the right-hand rule to decide whether the components of $\mathbf{a} \times \mathbf{b}$ are positive, negative, or 0.

x-component

y-component

z-component

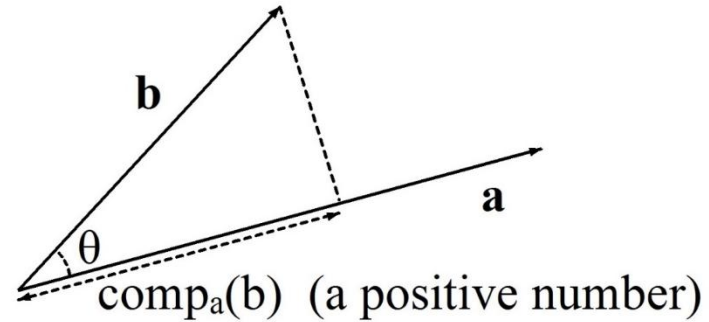
Visual: 3D Desmos - Cross Product Geometry

The magnitude of $\mathbf{a} \times \mathbf{b}$.

Thru algebra and using dot product rules (see book), it can be shown that

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$$

where θ is the smallest angle between $\mathbf{a}$ and $\mathbf{b}$. ($0 \leq \theta \leq \pi$)



Note: $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$
is the area of the parallelogram
formed by $\mathbf{a}$ and $\mathbf{b}$

Triangle Area / Normal to a Plane (HW 9)

Consider the points

$$P(2,0,2), Q(-2,1,4), R(3,2,6)$$

Find the area of the triangle PQR .

Visual: 3D Desmos

12.3/12.4 One-Page Summary

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Dot Product “10-star” fact:

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$, where θ is the tail-to-tail angle between $\mathbf{a}$ and $\mathbf{b}$. ($0 \leq \theta \leq \pi$)

Important consequence:

If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, then $\mathbf{a} \cdot \mathbf{b} = 0$.

Cross Product “10-star” fact:

$\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is orthogonal to *both* $\mathbf{a}$ and $\mathbf{b}$.

Another Cross-Product fact:

$|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$.

(and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$)